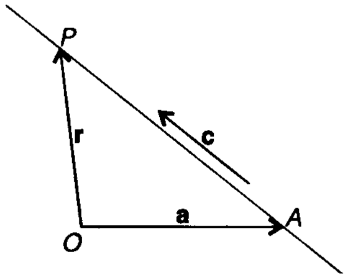
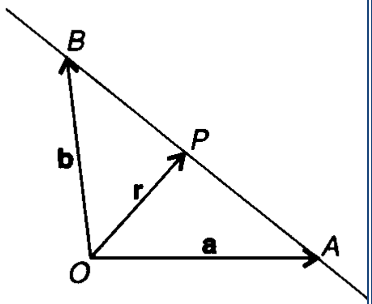
Day 4 Session 2

Vector Equation of a Straight Line



$\vec{r} = \vec{a} + t\vec{c}$, t : scalar parameter
$\vec{a}$: position vector of a fixed point on the straight line
$\vec{c}$: direction vector
$\vec{r}$: position vector of any point on straight line



Remark $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = (1-t)\vec{a} + t\vec{b}$

Exercise Find the vector equation of the straight line ℓ_1, in the direction of $i - 2j + 2k$ and passing through the point with position vector $(-1,2,-3)$.

Exercise Find the vector equation of the straight line through the point $(3,5,-4)$ in the direction of $i - j + k$. Find also the point on this line which has $4i$ as one component vector of its position vector.

Exercise Find the equation of the line joining the points $A(-1,2,-6)$ and $B(-4,8,3)$. Find the coordinates of the point of intersection of this line and the x-y plane and the ratio in which x-y plane divides AB.

Exercise Let $A = (8,-7,0)$ and $B = (2,-1,-3)$.

(a) Find the equation of the straight line AB.

(b) Find the perpendicular distance from the point $P(4,7,-9)$ to the line AB. Find also the foot of perpendicular.

Exercise The line joining two points $P(1,8,-1)$ and $Q(4,-4,2)$ meets the $xz-$ and

yz – planes

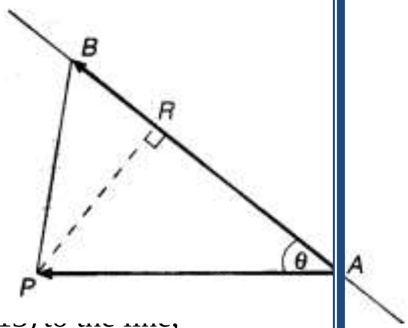
respectively at R and S. Find the coordinates of R and S and the ratios in which they

divide PQ.

Remark In above example (b), the distance from P to AB may also be found directly without calculating the foot of perpendicular. The method is outlined as follows:

By referring to Figure,

$$PR = AP\sin\theta = \frac{\left|\overrightarrow{AB}\times\overrightarrow{AP}\right|}{\left|\overrightarrow{AB}\right|}$$



Since

Exercise By finding the foot of perpendicular from the point $P(10,-1,13)$ to the line, $L: r = i + 5k + t(4i - 5j)$, find the equation of straight line passing through P and perpendicular to L, find the perpendicular distance from P to L.

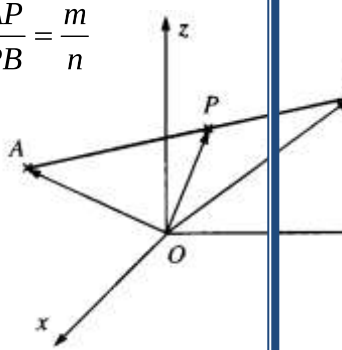
The Distance Between Two Points

Distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Section Formula

Let $P(x, y, z)$ divide the joint of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $\frac{AP}{PB} = \frac{m}{n}$

The Coordinate of the point P is $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$



Equations of Straight Lines

In vector form, the equation of straight line is $\vec{r} = \vec{a} + t\vec{c}$, where $\vec{r}$ is the position vector of any point in the line, $\vec{a}$ is fixed point on line and $\vec{c}$ is direction vector of line.

If $r = (x, y, z)$, $a = (x_1, y_1, z_1)$, $c = (a, b, c)$, we have

$$x\vec{i} + y\vec{j} + z\vec{k} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} + t(a\vec{i} + b\vec{j} + c\vec{k})$$

$$= (x_1 + ta)\vec{i} + (y_1 + tb)\vec{j} + (z_1 + tc)\vec{k}$$

Since $\vec{i}, \vec{j}, \vec{k}$ are basis vectors in R^3, we have

$$\begin{cases} x = x_1 + ta \\ y = y_1 + tb \\ z = z_1 + tc \end{cases} \quad \text{or} \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Parametric Form of a Straight Line

The equation of the straight line passing through the point (x_1, y_1, z_1) and with direction vector (a, b, c) can be expressed in the form of $\begin{cases} x = at + x_1 \\ y = bt + y_1 \\ z = ct + z_1 \end{cases}$ where t is a parameter.

This is called the *parametric form* of the straight line.

Symmetric Form of a Straight Line

The equation of the straight line passing through the point (x_1, y_1, z_1) and with direction vector (a, b, c) and is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

and this is called the *symmetric form* of the straight line.

General Form of a Straight Line

The equation of a straight line can be written as a linear system

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

which is called the *general form* of a straight line.

If given two points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, the equation of straight line becomes

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases} \quad \text{or} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Exercise Find the equation of the line joining the points $(2,0,3)$ and $(4,-1,2)$.

Exercise Find the equation of the line which passes through $(1,-3,2)$ and intersects the line

$$\frac{x-2}{1}=\frac{y}{3}=\frac{z+1}{-2}$$

Exercise ℓ_1 is a line passing through $A(1,0,-3)$ and $B(2,-4,-2)$, ℓ_2 is another line passing

through $C(2,-4,-1)$ and $D(6,-8,-3)$. Find, in degrees, the acute angle between ℓ_1 and ℓ_2.

Exercise Given two lines

$$L_1:\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

$$L_2:\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

the angle between two line is $\theta =$

Exercise Find the parametric form of a straight line L which passes through the point $(1,1,1)$ and parallel to the straight line $L_1:\frac{x-3}{2}=\frac{y-4}{-1}=\frac{z+2}{3}$. Show also that this line is perpendicular

to the straight line $L_2:\begin{cases} x = 3t+2 \\ y = 6t-1 \\ z = 4 \end{cases}$.

S 1

Let $L_1:\begin{cases} x = x_1 + \lambda_1 l_1 \\ y = y_1 + \lambda_1 m_1 \\ z = z_1 + \lambda_1 n_1 \end{cases}$ and $L_2:\begin{cases} x = x_2 + \lambda_2 l_2 \\ y = y_2 + \lambda_2 m_2 \\ z = z_2 + \lambda_2 n_2 \end{cases}$

To find the intersection point of line L_1 and L_2

we solve $\begin{cases} x_1 + \lambda_1 l_1 = x_2 + \lambda_2 l_2 \\ y_1 + \lambda_1 m_1 = y_2 + \lambda_2 m_2 \\ z_1 + \lambda_1 n_1 = z_2 + \lambda_2 n_2 \end{cases}$

i.e. find λ_1 and λ_2.

Note After finding λ_1 and λ_2 is any two equations, λ_1 and λ_2 must put into the 3rd equation in order to test whether it is satisfied or not.

Exercise Find the intersection point of the lines $\frac{x-4}{1}=\frac{y-5}{-1}=\frac{z-9}{-2}$ and $\frac{x-2}{1}=\frac{y}{-2}=\frac{4-z}{-1}$.

Exercise Find the intersection point of the lines $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-6}{3}$ and $\frac{x-4}{2}=\frac{y-6}{3}=\frac{z-11}{5}$.

S 2

Distance of a point $P(x_0, y_0, z_0)$ from the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

FIND P'.

Let P' be $(x_1+\lambda l, y_1+\lambda m, z_1+\lambda n)$.

Direction vector of PP' $(x_1+\lambda l-x_0, y_1+\lambda m-y_0, z_1+\lambda n-z_0)$

Direction vector of line (l, m, n)

$(x_1+\lambda l-x_0, y_1+\lambda m-y_0, z_1+\lambda n-z_0)\cdot(l, m, n)=0$

As λ is formed, P' can be determined and so $d=|PP'|$

Exercise Find the perpendicular distance from the point $P(4,7,-9)$ to the line $L:\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z+2}{1}$.

Exercise (a) Find the vector equation of the straight line ℓ_1, in the direction of $i-2j+2k$ and

passing through the point with position vector $(-1,2,-3)$.

(b) Find a vector parallel to the straight line ℓ_2 with vector equation

$$r=(-t+1)i+(2t-2)j+(3t+6)k$$

where t is a scalar parameter.

(c) Determine whether ℓ_1 meets ℓ_2; if so, find the point of intersection of ℓ_1 meets ℓ_2.

(d) Find the angle between ℓ_1 meets ℓ_2.

Exercise If the foot of the perpendicular from the point $P(4,7,-9)$ to the line

$$L: \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z+3}{1}$$

is Q, find the coordinates of Q. Hence, find the perpendicular distance from P to L.

Exercise Consider the two straight lines $L_1: \frac{x+1}{1} = \frac{y-2}{-2} = \frac{z+3}{2}$ and

$$L_2: \frac{x-1}{-1} = \frac{y+2}{2} = \frac{z-6}{3}.$$

Find the point of intersection of L_1 and L_2.

Find also the acute angle between L_1 and L_2.

Exercise Let $a = i + 3j - k$, $b = 3i + 6j$, $c = -2i + 4j - 3k$ be the position vectors of the points A,

B and C respectively.

(a) Find the equation of the line L, which passes through A and B.

(b) Find the shortest distance from C to L.

Theorem Given $L_1: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and

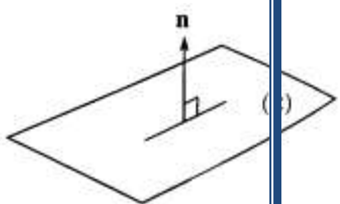
$$L_2: \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

$L_1 // L_2 \Leftrightarrow$ Their direction vectors are parallel $\Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Remark $L_1 \perp L_2 \Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

Plane and Equation of Plane

A vector perpendicular to (or orthogonal to) a plane is a normal vector to that plane. In Figure, $\vec{n}$ is a normal vector of the plane (π).



Normal vector of a plane is not unique, for if $\vec{n}$ is a normal vector, then $a\vec{n}$ (a is any non-zero real number) is also a normal vector.

Let $P_0(x_0, y_0, z_0)$ be a fixed point and $P(x, y, z)$ be any point on it.

Set $n = (A, B, C)$ i.e. A, B, C are given.

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$ ***(Vector Form)***

We have $(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$

$$\Rightarrow \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$ ***(Normal Form)***

Remark The general form of plane equation is $Ax + By + Cz + D = 0$.

Furthermore, if three points are given, $P_i(x_i, y_i, z_i)$ $i = 1,2,3$.

$$\begin{cases} Ax_1 & + By_1 & + Cz_1 & + D & = 0 \\ Ax_2 & + By_2 & + Cz_2 & + D & = 0 \\ Ax_3 & + By_3 & + Cz_3 & + D & = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} A(x - x_1) & + B(y - y_1) & + C(z - z_1) & = 0 \\ A(x_1 - x_2) & + B(y_1 - y_2) & + C(z_1 - z_2) & = 0 \\ A(x_2 - x_3) & + B(y_2 - y_3) & + C(z_2 - z_3) & = 0 \end{cases}$$

$$\Rightarrow \quad \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\because n = (A, B, C) \neq 0$ $\therefore$ The system has non-trivial solution of A, B, C.

Hence, $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$. It is an equation of plane. ***(3 Point Form)***

Exercise Find the equation of the plane passing through the points $P(2,4,3), Q(4,1,9)$ and $R(0,-1,6)$.

Find also its distance from the origin.

Exercise Find the equation of the plane passing through the point $(1,2,3)$ and parallel to the plane $x-3y+4z=3$. Find also its distance from the origin.

Exercise Find the equation of the plane containing the line $\frac{x-3}{1}=\frac{y-3}{2}=\frac{z}{1}$ and the origin.

Exercise Find the equation of the plane passing through the origin and the point $(3,-1,2)$ and parallel

to the line $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z+1}{1}$.

Exercise Find the equation of the plane containing the point $P(2,3,4)$ and perpendicular to both of the planes $\pi_1: 2x-y+2z-8=0$ and $\pi_2: x+2y-3z+7=0$.

Example Let $\prod$ be the plane containing $(2,1,0)$, $(1,0,1)$ and $(3,0,1)$. Suppose L is the straight line passing through $A(0,0,2)$ and perpendicular to $\prod$. Find

(a) the equation of $\prod$,

(b) the coordinates of the point of intersection of L and $\prod$,

(c) the distance from A to $\prod$.

Ans: (a) $y+z-1=0$ (b) $(0,-\frac{1}{2},\frac{3}{2})$ (c) $\frac{\sqrt{2}}{2}$

Exercise Find the equation of the plane which contains the origin and the line

$\frac{x-1}{2}=y-2=\frac{z-3}{-2}$.

Exercise Find the coordinates of the point at which the line joining the points (3,1,4) and $(-2,6,1)$ meets the plane $2x+y-3z=3$.

Exercise Find the equations of the line which contains the point (2,3,4) and is parallel to the line of intersection of the planes $x+y-2z=1$ and $2x-3y+z+3=0$.

Exercise Find the equation of the plane which contains the line $6x=3y-9=2z-10$ and is at right angles to the plane $2x+7y-3z=1$.

Exercise Find the equation of the plane containing the parallel lines

$$L_1 : \frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-2}{3} \quad \text{and} \quad L_2 : \frac{x}{1} = \frac{y-1}{-2} = \frac{z+3}{3}.$$

Exercise Find the equations of the following planes.

(a) Passing through the points $(3,1,0), (2,8,3)$ and $(1,3,-2)$.

(b) Having $x-$intercept = -3 and perpendicular to the line joining the points $(5,-1,-4)$ to $(-1,1,-7)$.l

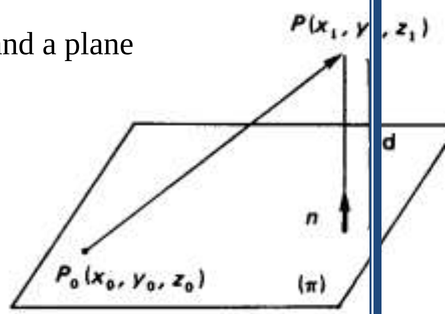
(c) Contains the line $x = y+2 = 6z-6$ and parallel to the line $x+1 = 2y+-12z.$

(d) Contains the lines

$$x = \frac{y-3}{2} = \frac{z-5}{3} \quad \text{and} \quad \frac{x+1}{2} = \frac{y-1}{5} = \frac{z-2}{3}.$$

Exercise Find a formula in order to find the distance from a fixed point $P(x_0, y_0, z_0)$ to the plane $Ax + By + Cz + D = 0$.

The perpendicular distance between a point and a plane



Theorem The perpendicular distance between a point $P(x_1, y_1, z_1)$ and a plane $\pi : Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Proof Let $P_0(x_0, y_0, z_0)$ be any point on the plane (π).

$Ai + Bj + Ck$ is a vector normal to the plane (π).

The unit vector n normal to the plane (π) is $n = \dfrac{Ai + Bj + Ck}{\sqrt{A^2 + B^2 + C^2}}$.

The perpendicular distance d between the point P and the plane is equal to the magnitude of the projection of $\overrightarrow{P_0P}$ on n.

Therefore $d = \left| \overrightarrow{P_0P} \cdot n \right|$

$$= \left| [(x_1 - x_0)i + (y_1 - y_0)j + (z_1 - z_0)k] \cdot \frac{Ai + Bj + Ck}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \frac{|A(x_1-x_0)+B(y_1-y_0)+C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_1+By_1+Cz_1-Ax_0-By_0-Cz_0|}{\sqrt{A^2+B^2+C^2}}$$

But, $D=-Ax_0-By_0-Cz_0$, since $P_0(x_0,y_0,z_0)$ lies on the plane.

$$\therefore \quad d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

Exercise Find the perpendicular distance between two parallel planes $(\pi_1): x-y+2z=6$ and $(\pi_2): 2x-2y+4z+5=0$.

Solution Take a point $P(0,0,3)$ on (π_1).

The required distance is just the perpendicular distance between P and (π_2).

i.e. $d = \dfrac{|2\times 0-2\times 0+4\times 3+5|}{\sqrt{2^2+(-2)^2+4^2}} = \dfrac{17}{12}\sqrt{6}$ units.

Exercise Find the equations of the two planes which are parallel to the plane

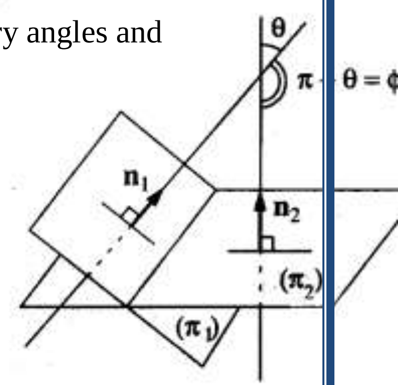
$(\pi): 3x-6y+2z+14=0$

and are 5 units away from the point $P(2,1,-3)$.

Angles Between Two planes

Given 2 planes $\pi_1: A_1x+B_1y+C_1z+D_1=0$ and $\pi_2: A_2x+B_2y+C_2z+D_2=0$

The angle between two planes is θ and ϕ, which are a pair of supplementary angles and



$$n_1\cdot n_2=|n_1||n_2|\cos\theta$$

$$\cos\theta = \frac{(A_1,B_1,C_1)\cdot(A_2,B_2,C_2)}{(\sqrt{A_1^2+B_1^2+C_1^2})(\sqrt{A_2^2+B_2^2+C_2^2})}$$

Remark (a) $\pi_1 // \pi_2 \Leftrightarrow n_1=t\,n_2$, $t: scalar$

$$\Leftrightarrow \frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=t$$

(b) $\pi_1 \perp \pi_2 \Leftrightarrow n_1\cdot n_2=0$

$$\Leftrightarrow A_1A_2+B_1B_2+C_1C_2=0$$

Equation of Plane Containing Two Given Lines

Given two lines $L_1 : \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$

$L_2 : \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

The normal vector of the required plane

$$n = (l_1, m_1, n_1) \times (l_2, m_2, n_2)$$

$$= \begin{vmatrix} i & j & k \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= (m_1 n_2 - m_2 n_1)\, i - (l_1 n_2 - l_2 n_1)\, j + (l_1 m_2 - l_2 m_1)\, k$$

$$n = (m_1 n_2 - m_2 n_1 \,, -l_1 n_2 + l_2 n_1 \,, l_1 m_2 - l_2 m_1)$$

$\therefore$ The equation of the plane

Exercise Find the equation of the plane containing two intersecting lines.

$$L_1 : \frac{x-2}{3} = \frac{y+1}{4} = \frac{z}{-2} \quad \text{and} \quad L_2 : \frac{x-2}{-1} = \frac{y+1}{3} = \frac{z}{2}$$

Solution

Exercise

Solve $\begin{cases} \mathbf{3x + 4y + 2z = 1} \\ \mathbf{6x + 2y + z = 0} \end{cases}$

From the above examples we conclude that the intersection of two planes is a line.

Alternatively,

consider $\mathbf{k = n_1 \times n_2}$

Family of Planes

Given two planes $\pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0$

$\pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0$

The family of planes is any plane containing the line of intersection π_1 and π_2.

$\pi : A_1 x + B_1 y + C_1 z + D_1 + k(A_2 x + B_2 y + C_2 z + D_2) = 0$, where k is a constant.

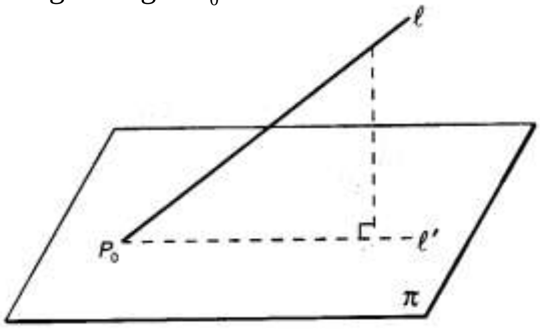
Exercise Find the equation of the plane containing the line $\begin{cases} x & -2y & +z & = & 4 \\ x & +6y & -5z & = & 0 \end{cases}$ and passing

the point $(-1,1,2)$.

Exercise Find the equation of the plane containing the line $L_1 : \begin{cases} x & +2z & = & 4 \\ y & -z & = & 8 \end{cases}$ and

parallel to the line $L_2 : \dfrac{x-3}{2} = \dfrac{y+4}{3} = \dfrac{z-7}{4}$.

Exercise (a) The position vector of a point $P(x, y, z)$ is given by $r = xi + yj + zk$.
In Figure, $P_0(x_0, y_0, z_0)$ is a point on the plane $\pi : r \cdot n = d$.
The line $\ell : r = r_0 + ta$, where t is a real scalar and $r_0 = x_0 i + y_0 j + z_0 k$, passing through P_0 and does not lie on π.



Show that the projection of ℓ on π is given by $r = r_0 + t\left(a - \dfrac{a \cdot n}{n \cdot n} n\right)$

where t is a real scalar.

(b) Consider the lines $\ell_1 : r = -3i + 6j + 2k + t(2i - 3j - k)$
and $\ell_2 : r = 10i - 19j - 2k + t(8i - 19j - 4k)$
and the plane $\pi : r \cdot (4i + j - 2k) = 4$

(i) Let A and B be the points at which π intersects ℓ_1 and ℓ_2 respectively.

Find the coordinates of A and B and show that AB is perpendicular to both ℓ_1 and ℓ_2.

(ii) Show that the projections of ℓ_1 and ℓ_2 on π are parallel.

Theorem Two given planes $\pi_1 : \dfrac{x - x_1}{A} = \dfrac{y - y_1}{B}$ and $\pi_2 : \dfrac{y - y_1}{B} = \dfrac{z - z_1}{C}$.

Prove that the equation of any plane through the line of intersection of π_1 and π_2

must contain a line $L : \dfrac{x - x_1}{A} = \dfrac{y - y_1}{B} = \dfrac{z - z_1}{C}$

Proof The equation of plane through the line of intersection of π_1 and π_2 is

$$B(x-x_1)-A(y-y_1)+k(C(y-y_1)-B(z-z_1)=0 \quad \cdots\cdots\cdots \ (*)$$

Normal Vector of (*) $n_1=(B,-A+kC,-Bk)$.

Direction vector of line L: $n_2=(A,B,C)$

$$n_1\cdot n_2=0$$

$\therefore$ (*) is parallel to line L.

Since (*) and L pass through the point (x_1,y_1,z_1).

$\therefore$ (*) contains L.

Coplanar Lines and Skew Lines

Coplanar Lines

Definition Two lines are said to be **Coplanar** if there exists a plane that contains both lines.

Two lines are **Coplanar** $\Leftrightarrow$ they must be either parallel or they intersect.

Theorem Two lines $(L_1):\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $(L_2):\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$

are coplanar if and only if $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0 \quad \cdots\cdots(*)$

\Exercise Show that the two lines

$$L_1:\frac{x-3}{2}=\frac{y-2}{-5}=\frac{z-1}{3} \text{ and } L_2:\frac{x-1}{-4}=\frac{y+2}{1}=\frac{z-6}{2}$$

are coplanar and intersect.

Exercise Show that the two lines

$L_1:\dfrac{x-2}{1}=\dfrac{y-1}{2}=\dfrac{z}{3}$ and $L_2:\dfrac{x-1}{4}=\dfrac{y-2}{1}=\dfrac{z-3}{-2}$ are coplanar.

Exercise If the lines $\dfrac{x-2}{1}=\dfrac{y-4}{p}=\dfrac{z-4}{1}$ and $\dfrac{x}{1}=\dfrac{y-3}{-1}=\dfrac{z-2}{q}$ are coplanar and perpendicular to each other, find p and q.

Exercise Show that the lines $\dfrac{x-2}{2}=\dfrac{y-3}{-1}=\dfrac{z+4}{3}$, $\dfrac{x-3}{1}=\dfrac{y+1}{3}=\dfrac{z-1}{-2}$ are coplanar.

Find the intersection and find the equation of the plane containing them.

Exercise (a) Show that the two lines $L_1 : \frac{x-1}{-1} = \frac{y+2}{2} = \frac{z-3}{1}$ and $L_2 : \frac{x+1}{3} = \frac{y-1}{2} = \frac{z+1}{-1}$ are non-coplanar.

(b) Find a straight line passing through the origin and intersecting each of the lines L_1 and L_2.

Skew Lines

Two straight lines are said to be **Skew** if they are non-coplanar i.e. neither do they intersect nor are they being parallel.

To find the shortest distance between them, we have to find the common perpendicular to both lines first. The method is illustrated by the following example.

Exercise It is given that the two lines

$$L_1 : \frac{x-5}{1} = \frac{y}{2} = \frac{z+1}{-1} \quad \text{and} \quad L_2 : \frac{x-2}{1} = \frac{y-4}{-1} = \frac{z}{1}$$

are non-coplanar. Find the shortest distance between them.

Exercise Solve $L_1 : \begin{cases} x + y = 0 \\ y + z = 0 \end{cases}$ and $L_2 : \begin{cases} x = 1 + 2t \\ y = 1 \\ z = 1 + t \end{cases}$.

Exercise Consider the line $L : \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z}{2}$ and the plane $\pi : x + y + z = 0$.

(a) Find the coordinates of the point where L intersects π.

(b) Find the angle between L and π.

Exercise Let L_1 be the line of intersection of the planes $x + y + z = 1$ and $x - y - z = 5$, and L_2 be the line of passing through $(1,1,-1)$ and intersecting L_1 at right line.

(a) Find a parametric equation of L_1.

(b) Find the coordinates of the point of intersection of L_1 and L_2, and a parametric equation of L_2

Exercise Find the image of the line $\frac{x+1}{2}=\frac{y-2}{1}=\frac{z-3}{-6}$ in the plane $3x+2y-5z=24$.

Exercise (a) Let m,n be vectors in R^3. Show that

(i) $\det\begin{pmatrix} m\cdot m & m\cdot n \\ m\cdot n & n\cdot n \end{pmatrix}=|m\times n|^2$

(ii) $(n\cdot n)m-(m\cdot n)n=n\times(m\times n)$

(b) Two planes $(r-a)\cdot m=0$ and $(r-b)\cdot n=0$ intersect in a Line L, where a,b,m,n are constant vectors and r is any position vector R^3. Express the real numbers λ and μ in terms of a,b,m and n such that the point represented by the position vector $p=\lambda m+\mu n$ lies on the line L.

Show that

$$p=(a\cdot m)\frac{n\times(m\times n)}{|m\times n|^2}+(b\cdot n)\frac{m\times(n\times m)}{|m\times n|^2}$$

Exercise Consider the line $L:\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z}{2}$ and the plane $\pi: x+y+z=0$.

(a) Find the coordinates of the point where L intersects π.

(b) Find the angle between L and π.

Ans: (a) $(-1,3,-2)$ (b) $\frac{\pi}{2}-\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Exercise Find the equations of the straight line which satisfies the following two conditions:

(i) passing through the point $(4,2,-3)$,

(ii) parallel to the planes $x+y+z-10=0$ and $x+2y=0$

Ans: $\frac{x-4}{-2}=\frac{y-2}{1}=\frac{z+3}{1}$

Exercise Find the equation of the plane passing through the line of intersection of the planes

$x+y+z-1=0$ and $x+4y+3z=0$

and parallel to the straight line $x-1=3y=3(z+1)$.

Ans: $-x+2y+z+2=0$

Exercise If the lines $\frac{x-2}{1}=\frac{y-4}{p}=\frac{z-4}{1}$ and $\frac{x}{1}=\frac{y-3}{-1}=\frac{z-2}{q}$

are coplanar and perpendicular to each other, find p and q.

Ans: $\begin{cases} p = 2 \\ q = 1 \end{cases}$ *or* $\begin{cases} p = \frac{1}{2} \\ q = -\frac{1}{2} \end{cases}$

Exercise Consider the lines

$$L_1 : \frac{x-2}{1} = \frac{y-3}{2} = \frac{z-3}{3} \quad \text{and} \quad L_2 : \frac{x-4}{2} = \frac{y-6}{3} = \frac{z-11}{5}$$

(a) Prove that L_1 and L_2 are non-coplanar .

(b) (i) Find the equation of the plane π containing L_1 and parallel to L_2 .

(ii) Find the equation of the plane π' containing L_2 and perpendicular to π .

(c) (i) Find the point S at which L_1 intersects π' .

(ii) Find the equations of line through S and perpendicular to both L_1 and L_2 .

Ans: (b) (i) $x + y - z - 2 = 0$ (ii) $8x - 7y + z - 1 = 0$

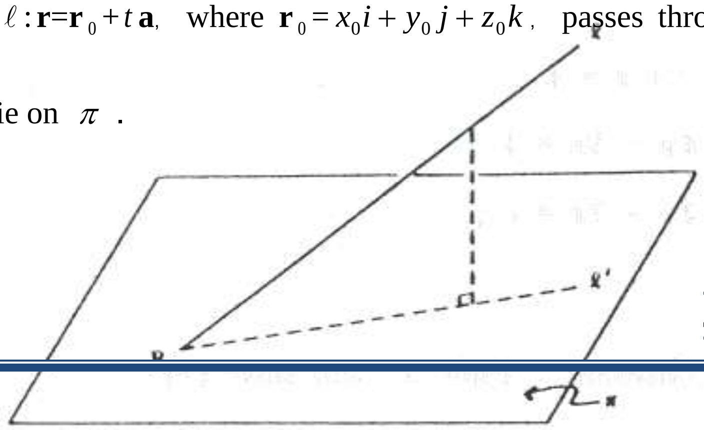
(c) (i) (1,1,0) (ii) $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-0}{-1}$

Exercise Find the equation of the plane containing the line $(L): \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{2}$ and the point $A(1,1,3)$

Ans: $2x + 3y - 6z = -13$

Exercise (a) The position vector of a point $R(x , y , z)$ is given by $z = xi + yj + zk$. In the figure, $R_0(x_0 , y_0 , z_0)$ is a point on the plane $\pi : \mathbf{r} . \mathbf{n} = \rho$.

The line $\ell : \mathbf{r} = \mathbf{r}_0 + t\,\mathbf{a}$, where $\mathbf{r}_0 = x_0 i + y_0 j + z_0 k$, passes through R_0 and does not lie on π .



Show that the projection of ℓ on π is given by $\ell' : \mathbf{r} = \mathbf{r}_0 + t\left(\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right)$, $t \in R$.

(b) Consider the lines $\ell_1 : \begin{cases} x = -1-2t \\ y = 3+3t \\ z = 1+t \end{cases}$, $t \in R$ and $\ell_2 : \begin{cases} x = 2-8t \\ y = 19t \\ z = 2+4t \end{cases}$, $t \in R$

and the plane $\pi_1 : 4x + y - 2z - 4 = 0$.

(i) Let P_1 and P_2 be the points at which π_1 intersects ℓ_1 and ℓ_2 respectively .

Find P_1 and P_2 and show that the line segment P_1P_2 is perpendicular to both ℓ_1 and ℓ_2 .

(ii) Show that the projections of ℓ_1 and ℓ_2 on π_1 are parallel .

Exercise (a) Let $\ell_1 : \begin{cases} x = a_1 + p_1 t \\ y = b_1 + q_1 t \\ z = c_1 + r_1 t \end{cases}$ *and* $\ell_2 : \begin{cases} x = a_2 + p_2 t \\ y = b_2 + q_2 t \\ z = c_2 + r_2 t \end{cases}$ be two given lines . Suppose ℓ_1

and ℓ_2 intersect .

(i) Show that $\begin{vmatrix} a_1 - a_2 & p_1 & p_2 \\ b_1 - b_2 & q_1 & q_2 \\ c_1 - c_2 & r_1 & r_2 \end{vmatrix} = 0$

(ii) If ℓ_1 and ℓ_2 are distinct, find a vector normal to the plane containing ℓ_1

and ℓ_2 .

Hence, or otherwise, obtain the equation of this plane .

(b) Consider the lines

$$L_1 : \begin{cases} x = pt \\ y = qt \\ z = rt \end{cases}, \quad L_2 : \begin{cases} x = qt \\ y = rt \\ z = pq \end{cases} \quad \text{and} \quad L_3 : \begin{cases} x = rt \\ y = pt \\ z = qt \end{cases}$$

where p , q and r are distinct and non-zero . Find the equation of a plane containing L_1 and perpendicular to the plane which contains L_2 and L_3 when

(i) $pq + qr + rp \neq 0$

(ii) $pq + qr + rp = 0$